

# Computing obstructions for existence of connections on modules

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## Abstract

We consider the notion of a connection on a module over a commutative ring, and recall the obstruction calculus for such connections. This obstruction calculus is defined using Hochschild cohomology. However, in order to compute with Gröbner bases, we need the conversion to a description using free resolutions. We describe our implementation in SINGULAR 3.0, available as the library `conn.lib`. Finally, we use the library to verify some known results and to obtain a new theorem for maximal Cohen-Macaulay (MCM) modules on isolated singularities. For a simple hypersurface singularity of dimension one or two, it is known that all MCM modules admit connections. We prove that for a simple threefold hypersurface singularity of type  $A_n$ ,  $D_n$  or  $E_n$ , only the free MCM modules admit connections if  $n \leq 50$ .

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## Introduction

Let  $k$  be an algebraically closed field of characteristic 0. For any commutative  $k$ -algebra  $A$  and any  $A$ -module  $M$ , we investigate whether there exist  $A$ -linear maps  $\nabla : \mathfrak{g} \rightarrow \text{End}_k(M)$  satisfying the Leibniz rule

$$\nabla_D(am) = a\nabla_D(m) + D(a)m \quad \text{for all } D \in \mathfrak{g}, a \in A, m \in M$$

on  $\mathfrak{g} = \text{Der}_k(A)$ , or possibly on a smaller subset  $\mathfrak{g} \subseteq \text{Der}_k(A)$  which is closed under the  $A$ -module and  $k$ -Lie algebra structures of  $\text{Der}_k(A)$ . We refer to such maps as  $\mathfrak{g}$ -connections, or connections in case  $\mathfrak{g} = \text{Der}_k(A)$ .

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In many situations, this question is related to the topology of the singularity  $X = \text{Spec}(A)$ . In fact, assume that  $k$  is the field of complex numbers, that  $X$  is an isolated singularity and that  $M$  is a maximal Cohen-Macaulay (MCM) module. Then  $M$  is locally free on the complement  $U \subset X$  of the singularity. If there is a connection  $\nabla : \text{Der}_k(A) \rightarrow \text{End}_k(M)$  that is also a Lie-algebra homomorphism, we get an integrable connection on the vector bundle  $\widetilde{M}|_U$  on  $U$ . Passing to the associated complex analytic manifold  $U^{an}$ , we get a representation of the fundamental group  $\pi_1(U^{an})$  via the Riemann-Hilbert correspondence, see for instance chapter 1 in Deligne (1970) for the general case, or Gustavsen and Ile (2006) for the case of normal surface singularities.

In this paper, we use algebraic methods to study the existence of  $\mathfrak{g}$ -connections on modules. To this aim, we recall the obstruction calculus for  $\mathfrak{g}$ -connections, which can be effectively implemented. We use Hochschild cohomology to define this obstruction theory, but for the implementation, the description of the obstructions in terms of free resolutions, given in section 4, is essential. We present our implementation as the library `conn.lib` (Eriksen and Gustavsen, 2006b) for the computer algebra system SINGULAR 3.0 (Greuel et al., 2005).

In the case of simple hypersurface singularities (of type  $A_n$ ,  $D_n$  or  $E_n$ ) in dimension  $d$ , there exists a connection on any MCM module if  $d \leq 2$ . Using our implementation, we get interesting results in higher dimensions: For  $d = 3$ , we show that the only MCM modules that admit connections are the free modules if  $n \leq 50$ , and experimental results indicate that the same result hold for  $d = 4$ .

These results led us to conjecture that for any simple hypersurface singularity of dimension  $d = 3$ , the only MCM modules that admit connections are the free modules. Using different techniques, we proved a more general result in Eriksen and Gustavsen (2006a): An MCM module over a simple hypersurface singularity of dimension  $d \geq 3$  admits a connection if and only if it is free.

In the case of simple elliptic surface singularities, it was shown in Kahn (1988), using analytic methods, that any MCM module admits a connection. We verify some instances of this result *algebraically*, using our implementation.

## 1 Basic definitions

Let  $k$  be an algebraically closed field of characteristic 0, and let  $A$  be a commutative  $k$ -algebra. A *Lie-Rinehart algebra* of  $A/k$  is a pair  $(\mathfrak{g}, \tau)$ , where  $\mathfrak{g}$  is an  $A$ -module and a  $k$ -Lie algebra, and  $\tau : \mathfrak{g} \rightarrow \text{Der}_k(A)$  is a morphism of

$A$ -modules and  $k$ -Lie algebras, such that

$$[D, aD'] = a[D, D'] + \tau_D(a)D'$$

for all  $D, D' \in \mathfrak{g}$  and all  $a \in A$ , see Rinehart (1963). A Lie-Rinehart algebra is the algebraic analogue of a *Lie algebroid*, and it is also known as a Lie pseudo-algebra or a Lie-Cartan pair.

When  $\mathfrak{g}$  is a subset of  $\text{Der}_k(A)$  and  $\tau : \mathfrak{g} \rightarrow \text{Der}_k(A)$  is the inclusion map, the pair  $(\mathfrak{g}, \tau)$  is a Lie-Rinehart algebra if and only if  $\mathfrak{g}$  is closed under the  $A$ -module and  $k$ -Lie algebra structures of  $\text{Der}_k(A)$ . We are mainly interested in Lie-Rinehart algebras of this type, and omit  $\tau$  from the notation.

Let  $\mathfrak{g}$  be a Lie-Rinehart algebra. For any  $A$ -module  $M$ , we define a  $\mathfrak{g}$ -connection on  $M$  to be an  $A$ -linear map  $\nabla : \mathfrak{g} \rightarrow \text{End}_k(M)$  such that

$$(1) \quad \nabla_D(am) = a\nabla_D(m) + D(a)m$$

for all  $D \in \mathfrak{g}$ ,  $a \in A$ ,  $m \in M$ . We say that  $\nabla$  satisfies the *derivation property* when condition (1) holds for all  $D \in \mathfrak{g}$ . If  $\nabla : \mathfrak{g} \rightarrow \text{End}_k(M)$  is a  $k$ -linear map that satisfies the derivation property, we call  $\nabla$  a  $k$ -linear  $\mathfrak{g}$ -connection on  $M$ . A *connection* on  $M$  is a  $\mathfrak{g}$ -connection on  $M$  with  $\mathfrak{g} = \text{Der}_k(A)$ .

Let  $\nabla$  be a  $\mathfrak{g}$ -connection on  $M$ . We define the *curvature* of  $\nabla$  to be the  $A$ -linear map  $R_\nabla : \mathfrak{g} \wedge \mathfrak{g} \rightarrow \text{End}_A(M)$  given by  $R_\nabla(D \wedge D') = [\nabla_D, \nabla_{D'}] - \nabla_{[D, D']}$  for all  $D, D' \in \mathfrak{g}$ . We say that  $\nabla$  is an *integrable  $\mathfrak{g}$ -connection* if  $R_\nabla = 0$ .

When  $A$  is a regular  $k$ -algebra, it is usual to define a *connection* on  $M$  to be a  $k$ -linear map  $\nabla : M \rightarrow M \otimes_A \Omega_A$  such that  $\nabla(am) = a\nabla(m) + m \otimes d(a)$  for all  $a \in A$ ,  $m \in M$ , see Katz (1970). Moreover, the *curvature* of  $\nabla$  is usually defined as the  $A$ -linear map  $R_\nabla : M \rightarrow M \otimes_A \Omega_A^2$  given by  $R_\nabla = \nabla^1 \circ \nabla$ , where  $\nabla^1$  is the natural extension of  $\nabla$  to  $M \otimes \Omega_A$ , and  $\nabla$  is an *integrable connection* if  $R_\nabla = 0$ .

Let  $A$  be any commutative  $k$ -algebra. For expository purposes, we define an  $\Omega$ -connection on an  $A$ -module  $M$  to be a connection on  $M$  in the sense of the preceding paragraph. By the universal property of  $\Omega_A$ , it follows that any (integrable)  $\Omega$ -connection on  $M$  induces an (integrable) connection on  $M$ .

When  $A$  is a regular  $k$ -algebra essentially of finite type, there is a bijective correspondence between connections on  $M$  and  $\Omega$ -connections on  $M$  for any finitely generated  $A$ -module  $M$ . However, there are many modules that admit connections but not  $\Omega$ -connections when  $A$  is a singular  $k$ -algebra, see subsection 6.3 for some examples.

## 2 Hochschild cohomology

Let  $k$  be an algebraically closed field of characteristic 0, and let  $A$  be any commutative  $k$ -algebra. For any  $A$ - $A$  bimodule  $Q$ , the *Hochschild cohomology*  $\mathrm{HH}^*(A, Q)$  of  $A$  with values in  $Q$  is the cohomology of the Hochschild complex  $(\mathrm{HC}^*, d^*)$ , where

$$\mathrm{HC}^n(A, Q) = \mathrm{Hom}_k(\otimes_k^n A, Q)$$

We recall that  $d^0 : Q \rightarrow \mathrm{Hom}_k(A, Q)$  is given by  $d^0(q)(a) = aq - qa$  for all  $a \in A$ ,  $q \in Q$  and that  $d^1 : \mathrm{Hom}_k(A, Q) \rightarrow \mathrm{Hom}_k(A \otimes_k A, Q)$  is given by  $d^1(\phi)(a \otimes b) = a\phi(b) - \phi(ab) + \phi(a)b$  for all  $a, b \in A$ ,  $q \in Q$ . For the definitions of the higher differentials, we refer to Weibel (1994). It is clear that  $\ker(d^1) = \mathrm{Der}_k(A, Q)$ , and we refer to  $\mathrm{im}(d^0)$  as the *trivial derivations*.

Given  $A$ -modules  $M, M'$ , we shall consider Hochschild cohomology of  $A$  with values in the bimodule  $Q = \mathrm{Hom}_k(M, M')$ . In this case, the natural exact sequence  $0 \rightarrow \mathrm{HH}^0(A, Q) \rightarrow \mathrm{HC}^0(A, Q) \rightarrow \ker(d^1) \rightarrow \mathrm{HH}^1(A, Q) \rightarrow 0$  has the form

$$(2) \quad 0 \rightarrow \mathrm{Hom}_A(M, M') \rightarrow \mathrm{Hom}_k(M, M') \xrightarrow{d^0} \rightarrow \mathrm{Der}_k(A, \mathrm{Hom}_k(M, M')) \rightarrow \mathrm{HH}^1(A, \mathrm{Hom}_k(M, M')) \rightarrow 0$$

It follows from Weibel (1994), theorem 8.7.10 and lemma 9.1.9, that there is a natural isomorphism of  $k$ -linear vector spaces

$$(3) \quad \mathrm{Ext}_A^n(M, M') \rightarrow \mathrm{HH}^n(A, \mathrm{Hom}_k(M, M'))$$

for any  $n \geq 0$ . For our purposes, it is useful to describe an identification of this type explicitly in the case  $n = 1$ .

Let  $(L_*, d_*)$  be a free resolution of  $M$ , and let  $\tau : M \rightarrow L_0$  be a  $k$ -linear section of the augmentation morphism  $\rho : L_0 \rightarrow M$ . Let moreover  $\phi : L_1 \rightarrow M'$  be a 1-cocycle in  $\mathrm{Hom}_A(L_*, M')$ . For any  $a \in A$ ,  $m \in M$ , we can find an element  $x = x(a, m) \in L_1$  such that  $d_0(x) = a\tau(m) - \tau(am)$ . It is easy to check that  $\psi(a)(m) = \phi(x)$  defines a derivation  $\psi \in \mathrm{Der}_k(A, \mathrm{Hom}_k(M, M'))$  which is independent of the choice of  $x$ .

**Lemma 1** *The assignment  $\phi \mapsto \psi$  defined above induces a natural injective map  $\sigma : \mathrm{Ext}_A^1(M, M') \rightarrow \mathrm{HH}^1(A, \mathrm{Hom}_k(M, M'))$  of  $k$ -linear vector spaces.*

**PROOF.** If  $\phi = \phi'd_0$  is a coboundary in  $\mathrm{Hom}_A(L_*, M')$ , then  $\psi$  is a trivial derivation, given by  $\psi(a)(m) = a\psi'(m) - \psi'(am)$  with  $\psi' = \phi'\tau$ . Hence the assignment induces a well-defined map  $\sigma$  of  $k$ -linear vector spaces. To see that it is injective, assume that  $\phi$  is a 1-cocycle in  $\mathrm{Hom}_A(L_*, M')$  which maps

to a trivial derivation  $\psi$ , given by  $\psi(a)(m) = a\psi'(m) - \psi'(am)$  for some  $\psi' \in \text{Hom}_k(M, M')$ . For any  $x \in L_0$ , there exists an element  $x' \in L_1$  such that  $d_0(x') = x - \tau\rho(x)$ , and  $x'$  is unique modulo  $\text{im}(d_1)$ . Let us define  $\phi' : L_0 \rightarrow M'$  to be the map given by  $\phi'(x) = \psi'\rho(x) + \phi(x')$ , this is clearly a well-defined map since  $\phi d_1 = 0$ . One may show that  $\phi'$  is  $A$ -linear, and it satisfies  $\phi' d_0 = \phi$  by construction. Hence  $\phi$  is a coboundary.

We do not claim that  $\sigma$  coincides with the identification (3) for  $n = 1$ . We shall use  $\sigma$  to obtain a concrete identification of certain classes defined using free resolutions, see section 4 for details, with the obstruction classes that we define in the next section. The important fact is therefore that  $\sigma$  is injective.

### 3 Obstruction theory

Let  $k$  be an algebraically closed field of characteristic 0, let  $A$  be a commutative  $k$ -algebra, and let  $M$  be an  $A$ -module.

**Proposition 2** *There is a canonical class  $\mathbf{a}(M) \in \text{Ext}_A^1(M, M \otimes_A \Omega_A)$ , called the Atiyah class of  $M$ , such that  $\mathbf{a}(M) = 0$  if and only if there is an  $\Omega$ -connection on  $M$ . In this case, there is a transitive and effective action of  $\text{Hom}_A(M, M \otimes_A \Omega_A)$  on the set of  $\Omega$ -connections on  $M$ .*

**PROOF.** We consider the derivation  $\psi : A \rightarrow \text{Hom}_k(M, M \otimes_A \Omega_A)$  given by  $\psi(a)(m) = m \otimes d(a)$ , where  $d$  is the universal derivation of  $A$ , and define  $\mathbf{a}(M)$  to be the class in  $\text{Ext}_A^1(M, M \otimes_A \Omega_A)$  corresponding to the class  $[\psi]$  of  $\psi$  in  $\text{HH}^1(A, \text{Hom}_k(M, M \otimes_A \Omega_A))$  via the identification (3). Using the sequence (2), the proposition follows easily.

**Proposition 3** *There is a canonical map  $g : \text{Der}_k(A) \rightarrow \text{Ext}_A^1(M, M)$ , called the Kodaira-Spencer map of  $M$ , with the following properties:*

- (1) *The Kodaira-Spencer kernel  $\mathbf{V}(M) = \ker(g)$  is a Lie algebroid of  $A/k$ ,*
- (2) *For any  $D \in \text{Der}_k(A)$ , there exists an operator  $\nabla_D \in \text{End}_k(M)$  with derivation property with respect to  $D$  if and only if  $D \in \mathbf{V}(M)$ .*

*In particular,  $\mathbf{V}(M)$  is maximal among the subsets  $\mathfrak{g} \subseteq \text{Der}_k(A)$  such that there exists a  $k$ -linear  $\mathfrak{g}$ -connection on  $M$ .*

**PROOF.** For any  $D \in \text{Der}_k(A)$ , we consider  $\psi_D \in \text{Der}_k(A, \text{End}_k(M))$  given by  $\psi_D(a)(m) = D(a)m$ , and denote by  $g(D)$  the class in  $\text{Ext}_A^1(M, M)$  corresponding to the class  $[\psi_D]$  of  $\psi_D$  in  $\text{HH}^1(A, \text{End}_k(M))$  via the identification

(3). This defines the Kodaira-Spencer map  $g$  of  $M$ , which is  $A$ -linear by definition. Clearly, its kernel  $\mathbf{V}(M)$  is closed under the Lie product. Using the exact sequence (2), it easily follows that there exists an operator  $\nabla_D$  with derivation property with respect to  $D$  if and only if  $D \in \mathbf{V}(M)$ .

We remark that the Kodaira-Spencer map  $g : \text{Der}_k(A) \rightarrow \text{Ext}_A^1(M, M)$  is the contraction against the Atiyah class  $\mathbf{a}(M) \in \text{Ext}_A^1(M, M \otimes_A \Omega_A)$ . See also Källström (2005), section 2.2 for another proof of proposition 3.

**Proposition 4** *There is a canonical class  $\text{lc}(M) \in \text{Ext}_A^1(\mathbf{V}(M), \text{End}_A(M))$  such that  $\text{lc}(M) = 0$  if and only if there exists a  $\mathbf{V}(M)$ -connection on  $M$ . In this case, there is a transitive and effective action of  $\text{Hom}_A(\mathbf{V}(M), \text{End}_A(M))$  on the set of  $\mathbf{V}(M)$ -connections on  $M$ .*

**PROOF.** Let  $\mathbf{V} = \mathbf{V}(M)$ , choose a  $k$ -linear  $\mathbf{V}$ -connection  $\nabla : \mathbf{V} \rightarrow \text{End}_k(M)$  on  $M$ , and let  $\phi \in \text{Der}_k(A, \text{Hom}_k(\mathbf{V}, \text{End}_A(M)))$  be the derivation given by  $\phi(a)(D) = a\nabla_D - \nabla_{aD}$ . We denote by  $\text{lc}(M)$  the class in  $\text{Ext}_A^1(\mathbf{V}, \text{End}_A(M))$  corresponding to the class  $[\phi]$  of  $\phi$  in  $\text{HH}^1(A, \text{Hom}_k(\mathbf{V}, \text{End}_A(M)))$  via the identification (3). One may check that this class is independent of  $\nabla$ . Using the exact sequence (2), the proposition follows easily.

There is a natural short exact sequence  $0 \rightarrow \text{End}_A(M) \rightarrow c(M) \rightarrow \mathbf{V}(M) \rightarrow 0$  of left  $A$ -modules, where  $c(M) = \{P \in \text{End}_k(M) : [P, a] \in A \text{ for all } a \in A\}$  is the module of first order differential operators on  $M$  with scalar symbol, and  $c(M) \rightarrow \text{Der}_k(A)$  is the natural map, given by  $P \mapsto [P, -]$ , with image  $\mathbf{V}(M)$ , see also Källström (2005), proposition 2.2.10. We remark that this extension of left  $A$ -modules splits if and only if  $\text{lc}(M) = 0$ .

**Lemma 5** *For any  $A$ -modules  $M, M'$ , we have  $\mathbf{V}(M \oplus M') = \mathbf{V}(M) \cap \mathbf{V}(M')$ , and  $\text{lc}(M \oplus M') = 0$  if and only if  $\text{lc}(M) = \text{lc}(M') = 0$ .*

Lemma 5 is a direct consequence of proposition 3 and 4. We remark that the first part also follows from Buchweitz and Liu (2004), lemma 3.4.

## 4 Computing with free resolutions

We define new classes  $\mathbf{a}(M)' \in \text{Ext}_A^1(M, M \otimes_A \Omega_A)$ ,  $g'(D) \in \text{Ext}_A^1(M, M)$  for all  $D \in \text{Der}_k(A)$ , and  $\text{lc}(M)' \in \text{Ext}_A^1(\mathbf{V}(M), \text{End}_A(M))$  in this section, and show that via  $\sigma$ , these classes correspond to the obstructions  $\mathbf{a}(M)$ ,  $g(D)$  and  $\text{lc}(M)$  defined in section 3. Since the new classes are defined using free resolutions, we can use Gröbner bases to compute them in SINGULAR.

Let  $k$  be an algebraically closed field of characteristic 0, let  $A$  be a commutative  $k$ -algebra essentially of finite type, and let  $M$  be a finitely generated  $A$ -module. Let

$$0 \leftarrow M \xleftarrow{\rho} L_0 \xleftarrow{d_0} L_1 \leftarrow \dots$$

be a free resolution of  $M$  such that  $L_i$  has finite rank for all  $i \geq 0$ . We choose bases  $\{e_1, \dots, e_m\}$  of  $L_0$  and  $\{f_1, \dots, f_n\}$  of  $L_1$ , and write  $(a_{ij})$  for the matrix of  $d_0 : L_1 \rightarrow L_0$  with respect to the chosen bases. One may show that the matrix  $(d(a_{ij}))$ , considered as an  $A$ -linear map  $L_1 \rightarrow L_0 \otimes_A \Omega_A$ , defines a 1-cocycle in  $\text{Hom}_A(L_*, M \otimes_A \Omega_A)$  and therefore a cohomology class  $\mathfrak{a}(M)' \in \text{Ext}_A^1(M, M \otimes_A \Omega_A)$ . The class  $\mathfrak{a}(M)'$  is called the *Atiyah class* of  $M$ , see Angéniol and Lejeune-Jalabert (1989).

For any  $D \in \text{Der}_k(A)$ , the matrix  $(D(a_{ij}))$ , considered as an  $A$ -linear map  $L_1 \rightarrow L_0$ , defines a 1-cocycle in  $\text{Hom}_A(L_*, M)$  and therefore a cohomology class  $g'(D) \in \text{Ext}_A^1(M, M)$ . We remark that the map  $D \mapsto g'(D)$  is the contraction against  $\mathfrak{a}(M)'$ , and it is therefore well-known that  $g'$  is the Kodaira-Spencer map of  $M$ , see Illusie (1971).

Since  $A$  is essentially of finite type over  $k$ , it follows that  $\mathbf{V}(M)$  is a left  $A$ -module of finite type. Let

$$0 \leftarrow \mathbf{V}(M) \xleftarrow{\rho'} L'_0 \xleftarrow{d'_0} L'_1 \leftarrow \dots$$

be a free resolution of  $\mathbf{V}(M)$  such that  $L'_i$  has finite rank for all  $i \geq 0$ . We choose bases  $\{e'_1, \dots, e'_p\}$  of  $L'_0$  and  $\{f'_1, \dots, f'_q\}$  of  $L'_1$ , and write  $(c_{ij})$  for the matrix of  $d'_0 : L'_1 \rightarrow L'_0$  with respect to the chosen bases. For  $1 \leq i \leq p$ , let us write  $D_i = \rho'(e'_i)$ . Since  $D_i \in \mathbf{V}(M)$ , we can find a  $k$ -linear operator  $\nabla_i : M \rightarrow M$  which has the derivation property with respect to  $D_i$  by proposition 3. We allow a slight abuse of notation, and write  $D_i : L_0 \rightarrow L_0$  for the action given by  $D_i(ae_j) = D(a)e_j$  for all  $a \in A$  and  $1 \leq j \leq n$ . Since the map

$$\nabla_i \rho - \rho D_i : L_0 \rightarrow M$$

is  $A$ -linear, we can lift it to an  $A$ -linear map  $P_i : L_0 \rightarrow L_0$  via  $\rho$ , i.e. such that  $\nabla_i \rho - \rho D_i = \rho P_i$ , and this implies that  $\nabla_i \rho = \rho(D_i + P_i)$ .

Let us consider  $R_j = \sum_i c_{ij} P_i$  as an  $A$ -linear map  $R_j : L_0 \rightarrow L_0$  for  $1 \leq j \leq q$ . Since  $\sum_i c_{ij} D_i = 0$  for all  $j$ , we see that  $R_j$  induces an  $A$ -linear operator on  $M$  for  $1 \leq j \leq q$ . One may show that the  $A$ -linear map  $L'_1 \rightarrow \text{End}_A(L_0)$  given by  $f'_j \mapsto R_j$  induces a 1-cocycle in  $\text{Hom}_A(L'_*, \text{End}_A(M))$ , and therefore defines a cohomology class  $\text{lc}(M)'$  in  $\text{Ext}_A^1(\mathbf{V}(M), \text{End}_A(M))$ .

**Lemma 6** *Let  $\mathfrak{a}(M)$ ,  $g(D)$  and  $\text{lc}(M)$  be the obstructions defined in section 3. Then we have:*

- (1)  $\mathfrak{a}(M) = 0$  if and only if  $\mathfrak{a}(M)' = 0$ ,

- (2)  $\ker(g) = \ker(g')$ ,  
(3)  $\mathrm{lc}(M) = 0$  if and only if  $\mathrm{lc}(M)' = 0$ .

**PROOF.** To prove the first two parts, it is enough to show that  $\mathbf{a}(M)$  and  $\mathbf{a}(M)'$  maps to the same element in  $\mathrm{HH}^1(A, \mathrm{Hom}_k(M, M \otimes_A \Omega_A))$ , since  $\sigma$  is functorial,  $g$  is the contraction against  $\mathbf{a}(M)$ , and  $g'$  is the contraction against  $\mathbf{a}(M)'$ . Since  $\mathbf{a}(M)$  maps to  $[\psi] \in \mathrm{HH}^1(A, \mathrm{Hom}_k(M, M \otimes_A \Omega_A))$ , the class of the derivation  $\psi$  defined in the proof of proposition 2, we must show that  $\sigma(\mathbf{a}(M)') = [\psi]$ . This follows from the fact that  $(d(a_{ij})) = dd_0 - d_0d$ , where  $d : L_i \rightarrow L_i \otimes_A \Omega_A$  is the natural action of the universal derivation  $d$  on  $L_i$  for  $i = 0, 1$  with respect to the chosen bases.

To prove the third part, it is enough to show that  $\mathrm{lc}(M)$  and  $\mathrm{lc}(M)'$  maps to the same element in  $\mathrm{HH}^1(A, \mathrm{Hom}_k(\mathbf{V}(M), \mathrm{End}_A(M)))$ . Since  $\mathrm{lc}(M)$  maps to  $[\phi] \in \mathrm{HH}^1(A, \mathrm{Hom}_k(\mathbf{V}(M), \mathrm{End}_A(M)))$ , the class of the derivation  $\phi$  defined in the proof of proposition 4, we must show that  $\sigma(\mathrm{lc}(M)') = [\phi]$ . Let us consider the  $A$ -linear map  $\nabla : L'_0 \rightarrow \mathrm{End}_k(M)$  given by  $e'_i \mapsto \nabla_i$  for  $1 \leq i \leq p$ , and notice that  $\nabla d'_0(f'_j) = R_j$  for  $1 \leq j \leq q$  and that  $\nabla \tau'$  is a  $k$ -linear  $\mathbf{V}(M)$ -connection on  $M$  for any  $k$ -linear section  $\tau'$  of  $\rho'$ . For any  $a \in A$ ,  $D \in \mathbf{V}(M)$ , choose  $x \in L'_1$  such that  $d'_0(x) = a\tau'(D) - \tau'(aD)$ . Then the cocycle  $f'_j \mapsto R_j$  maps  $x$  to  $\nabla(a\tau'(D) - \tau'(aD)) = a\nabla\tau'(D) - \nabla\tau'(aD)$ . By the definition of  $\sigma$ , it follows that  $\sigma(\mathrm{lc}(M)') = [\phi]$ .

We remark that the first and second part of lemma 6 follows from more general results, see for instance Buchweitz and Flenner (2003).

## 5 Implementation

Let  $k$  be an algebraically closed field of characteristic 0, let  $A$  be a  $k$ -algebra essentially of finite type, and let  $M$  be a finitely generated  $A$ -module. In this section, we explain the procedures `Der()`, `AClass(M)`, `KSKernel(M)` and `LClass(M)` in the SINGULAR library `conn.lib` (Eriksen and Gustavsen, 2006b), which calculate the derivation module  $\mathrm{Der}_k(A)$ , the Atiyah class  $\mathbf{a}(M)'$ , the Kodaira-Spencer kernel  $\mathbf{V}(M)$  and the class  $\mathrm{lc}(M)'$ . We use the notation from section 4.

In the implementation, we identify elements of free  $A$ -modules (with given bases) with column vectors, and identify  $A$ -linear maps of free  $A$ -modules (with given bases) with left multiplication by matrices. We use presentation matrices to represent modules, and assume that  $A$  is the base ring in SINGULAR.



**Der()**: Let us recall how to compute  $\text{Der}_k(A)$ . If  $A$  is of finite type over  $k$ , we may assume that  $A = S/I$ , where  $S = k[x_1, \dots, x_n]$  and  $I = (F_1, \dots, F_m) \subseteq S$  is an ideal. The Jacobian matrix  $J = (\partial F_i / \partial x_j)$  defines an  $A$ -linear map  $J : A^n \rightarrow A^m$ , and it is well-known that  $\mathbf{a} \mapsto a_1 \partial / \partial x_1 + \dots + a_n \partial / \partial x_n$  defines an isomorphism  $\ker(J) \rightarrow \text{Der}_k(A)$  of left  $A$ -modules. Finally, if  $B$  is a localization of  $A$ , then  $\text{Der}_k(B) \cong B \otimes_A \text{Der}_k(A)$ , so  $\text{Der}_k(B) \cong \ker(B \otimes_A J)$ . In the implementation, we write **j** for the matrix  $J$  (respectively  $B \otimes_A J$ ), and **d** for the matrix with column space  $\ker(J)$  (respectively  $\ker(B \otimes_A J)$ ). **Der()** returns the matrix **d**.

**AClass(M)**: In section 4, we have seen that  $\mathbf{a}(M)' \in \text{Ext}_A^1(M, M \otimes_A \Omega_A)$  is represented by  $(d(a_{ij})) : L_1 \rightarrow L_0 \otimes_A \Omega_A$ , where  $d : A \rightarrow \Omega_A$  is the universal derivation, and  $(a_{ij})$  is the matrix of  $d_0$  with respect to chosen bases of  $L_0$  and  $L_1$ . Choose a free resolution

$$0 \leftarrow \Omega_A \xleftarrow{\rho''} L_0'' \xleftarrow{d_0''} L_1'' \leftarrow \dots$$

of  $\Omega_A$ , and lift  $(d(a_{ij}))$  to an  $A$ -linear map  $\Delta : L_1 \rightarrow L_0 \otimes_A L_0''$ . It follows that  $\mathbf{a}(M)' = 0$  if and only if  $\Delta \in H_0$ , where  $H_0 \subseteq \text{Hom}_A(L_1, L_0 \otimes_A L_0'')$  is the submodule given by  $H_0 = \text{im}(\text{Hom}_A(d_0, L_0 \otimes_A L_0'')) + \text{im}(\text{Hom}_A(L_1, \delta_0))$  and  $\delta_0 = d_0 \otimes \text{id} - \text{id} \otimes d_0'' : (L_1 \otimes_A L_0'') \oplus (L_0 \otimes_A L_1'') \rightarrow L_0 \otimes_A L_0''$  is the differential in the complex  $L_* \otimes_A L_*''$ . In the implementation, we write **kPres** for the matrix of  $\delta_0$ , **rel** for the matrix with column space  $H_0$ , and **cm** for the representative of  $\Delta$  with respect to suitable bases. **AClass(M)** returns  $(\mathbf{cm} \notin \mathbf{rel})$ .

**KSKernel(M)**: For any  $D \in \text{Der}_k(A)$ , we have seen that  $g'(D) \in \text{Ext}_A^1(M, M)$  is represented by  $(D(a_{ij})) : L_1 \rightarrow L_0$ , where  $(a_{ij})$  is the matrix of  $d_0$  with respect to chosen bases of  $L_0$  and  $L_1$ . We write  $\Delta : \text{Der}_k(A) \rightarrow \text{Hom}_A(L_1, L_0)$  for the  $A$ -linear map given by  $D \mapsto (D(a_{ij}))$ . It follows that  $g'(D) = 0$  if and only if  $\Delta(D) \in H_0$ , where  $H_0 \subseteq \text{Hom}_A(L_1, L_0)$  is the submodule given by  $H_0 = \text{im}(\text{Hom}_A(d_0, L_0)) + \text{im}(\text{Hom}_A(L_1, d_0))$ . Choose a free resolution

$$0 \leftarrow \text{Der}_k(A) \xleftarrow{\rho'''} L_0''' \xleftarrow{d_0'''} L_1''' \leftarrow \dots$$

of  $\text{Der}_k(A)$ , and lift  $\Delta$  to an  $A$ -linear map  $\gamma : L_0''' \rightarrow \text{Hom}_A(L_1, L_0)$ . It follows that  $\mathbf{V}(M) = \rho'''(\gamma^{-1}(H_0)) \subseteq \text{Der}_k(A)$ . In the implementation, we write **rel** for the matrix with column space  $H_0$ , **g** for the matrix of  $\gamma$  and **ker** for the matrix with column space  $\gamma^{-1}(H_0)$  with respect to suitable bases. **KSKernel(M)** returns  $(\text{Der}() \cdot \mathbf{ker} \neq \text{Der}())$ .

**LClass(M)**: There is a  $k$ -linear connection  $\nabla : \mathbf{V} \rightarrow \text{End}_k(M)$  on  $\mathbf{V} = \mathbf{V}(M)$  by proposition 3, and  $\nabla_{D_i}$  can be lifted to a  $k$ -linear operator  $D_i + P_i : L_0 \rightarrow L_0$  for some  $P_i \in \text{End}_A(L_0)$  for  $1 \leq i \leq p$ , where we write  $D_i = \rho'(e'_i)$ . Let  $\Lambda : L_0' \rightarrow \text{Hom}_A(L_0, L_0)$  be the  $A$ -linear map given by  $e'_i \mapsto P_i$ . Then  $\Lambda d_0'$  induces an  $A$ -linear map  $L_1' \rightarrow \text{End}_A(M)$ , since any  $\phi \in \text{im}(\Lambda d_0')$  satisfies

$\phi d_0 = d_0 \psi$  for some  $\psi \in \text{Hom}_A(L_1, L_1)$ , and we have seen that  $\Lambda d'_0$  represents  $\text{lc}(M)' \in \text{Ext}_A^1(\mathbf{V}, \text{End}_A(M))$ . Choose a free resolution

$$0 \leftarrow \text{End}_A(M) \xleftarrow{\rho''''} L_0'''' \xleftarrow{d_0''''} L_1'''' \leftarrow \dots$$

of  $\text{End}_A(M)$ , and lift  $\Lambda d'_0$  to an  $A$ -linear map  $\lambda : L_1' \rightarrow L_0''''$ . It follows that  $\text{lc}(M)' = 0$  if and only if  $\lambda \in H_0$ , where  $H_0 \subseteq \text{Hom}_A(L_1', L_0''')$  is the submodule  $H_0 = \text{im}(\text{Hom}_A(d'_0, L_0''')) + \text{im}(\text{Hom}_A(L_1', d_0'''))$ . In the implementation, we write `vRel` for the matrix with column space  $H_0$ , `L` for the matrix of  $\Lambda$ , and `lc` for the representative of  $\lambda$  with respect to suitable bases. `LClass(M)` returns `(lc ∉ vRel)`.

## 6 Results on maximal Cohen-Macaulay modules

Let  $k$  be an algebraically closed field of characteristic 0, and let  $(A, m)$  be a local complete commutative Noetherian  $k$ -algebra. We say that a finitely generated  $A$ -module  $M$  is a maximal Cohen-Macaulay (MCM) module if  $\text{depth}(M) = \dim(A)$ . In this section, we give some results on the existence of connections on MCM modules over isolated singularities.

The SINGULAR library `conn.lib` (Eriksen and Gustavsen, 2006b) can be used to calculate the obstructions  $\mathbf{V}(M)$  and  $\text{lc}(M)$  for the existence of connections on an  $A$ -module  $M$  when  $A$  is a  $k$ -algebra of finite type or a local  $k$ -algebra essentially of finite type, as explained in section 5, using the monomial ordering `dp` (or `lp`) respectively `ds`. However, there is no direct way of computing with complete algebras in SINGULAR. Nevertheless, thanks to the following result, all examples in this section can be verified using the library `conn.lib`:

**Lemma 7** *Let  $A$  be a commutative  $k$ -algebra of finite type with a distinguished maximal ideal  $m$ , and let  $M$  be a finitely generated  $A$ -module. Consider  $M_m$  as a module over the local ring  $A_m$  and  $\widehat{M}$  as a module over the  $m$ -adic completion  $\widehat{A}$ . If  $M$  is locally free on  $\text{Spec } A \setminus \{m\}$ , then*

- (1)  $\mathbf{V}(M) = \text{Der}_k(A) \Leftrightarrow \mathbf{V}(M_m) = \text{Der}_k(A_m) \Leftrightarrow \mathbf{V}(\widehat{M}) = \text{Der}_k(\widehat{A})$ ,
- (2)  $\text{lc}(M) = 0 \Leftrightarrow \text{lc}(M_m) = 0 \Leftrightarrow \text{lc}(\widehat{M}) = 0$ .

### 6.1 Simple hypersurface singularities

Let  $A$  be the complete local ring of a simple hypersurface singularity, see Arnol'd (1981) and Wall (1984), and let  $d \geq 1$  be the dimension of  $A$ . Then

$A \cong k[[x, y, z_1, \dots, z_{d-1}]]/(f)$ , where  $f$  is of the form

$$\begin{aligned} A_n : f &= x^2 + y^{n+1} + z_1^2 + \dots + z_{d-1}^2 & n &\geq 1 \\ D_n : f &= x^2 y + y^{n-1} + z_1^2 + \dots + z_{d-1}^2 & n &\geq 4 \\ E_6 : f &= x^3 + y^4 + z_1^2 + \dots + z_{d-1}^2 \\ E_7 : f &= x^3 + xy^3 + z_1^2 + \dots + z_{d-1}^2 \\ E_8 : f &= x^3 + y^5 + z_1^2 + \dots + z_{d-1}^2 \end{aligned}$$

It is known that the simple hypersurface singularities are exactly the hypersurface singularities which are of finite CM representation type, i.e. which have a finite number of isomorphism classes of indecomposable MCM  $A$ -modules, see Knörrer (1987) and Buchweitz et al. (1987). The MCM modules are given by matrix factorizations, see Eisenbud (1980), and are completely classified. A complete list can be found in Greuel and Knörrer (1985) in the curve case and obtained from the McKay correspondence in the surface case. In higher dimensions, this list can be obtained using Knörrer's periodicity theorem, see Knörrer (1987) and Schreyer (1987). See also Yoshino (1990) for an overview.

We mention the following result for completeness. In the curve case, the result appears in Eriksen (2006) and all computations were originally done by hand constructing a connection on each indecomposable MCM module. Now we can check the result for  $E_6$ ,  $E_7$ ,  $E_8$  and for  $A_n$ ,  $D_n$  for  $n \leq 50$  using SINGULAR. In the surfaces case, the result is known to specialists, but has not been published as far as we know. The simple hypersurface singularities are the quotient surface singularities that are hypersurfaces. Then the result follows from an observation of J. Christophersen, see Eriksen and Gustavsen (2006a) or Maakestad (2005) for details.

**Theorem 8** *Let  $A$  be the complete local ring of a simple hypersurface singularity (of type  $A_n$ ,  $D_n$  or  $E_n$ ) of dimension  $d \leq 2$ . Then there is a connection on any MCM  $A$ -module.*

In contrast to the situation in dimension one and two, experimental data suggest that there are very few MCM modules over simple singularities of dimension  $d \geq 3$  that admit connections. We have discovered the following result using SINGULAR:

**Theorem 9** *Let  $A$  be the complete local ring of a simple hypersurface singularity (of type  $A_n$ ,  $D_n$  or  $E_n$ ) of dimension  $d = 3$ , and let  $M$  be an MCM  $A$ -module. If  $n \leq 50$ , then there is a connection on  $M$  if and only if  $M$  is free.*

Experimental results indicate that the same result hold in dimension  $d = 4$ . Due to the computational cost, we have not been able to calculate  $\text{lc}(M)$  for any non-free MCM module  $M$  over simple singularities in dimension  $d \geq 5$  using our implementation.

These results led us to conjecture that for any simple hypersurface singularity of dimension  $d = 3$ , the only MCM modules that admit connections are the free modules. Using different techniques, we proved a more general result in Eriksen and Gustavsen (2006a): An MCM module over a simple hypersurface singularity of dimension  $d \geq 3$  admits a connection if and only if it is free.

## 6.2 Simple elliptic surface singularities

Let  $k = \mathbb{C}$  and let  $A$  be the complete local ring of a simple elliptic surface singularity over  $k$ . Then  $A$  is of tame CM representation type, and all indecomposable MCM  $A$ -modules have been classified, see Kahn (1989). Moreover, it follows from the results in Kahn (1988) that any MCM  $A$ -module admits a connection. Kahn obtained these results by analytic methods.

We remark that we can prove many instances of Kahn's result *algebraically* using the library `conn.lib`. For instance, when  $A = k[[x, y, z]]/(x^3 + y^3 + z^3)$ , we can use presentation matrices of MCM  $A$ -modules from Laza et al. (2002).

## 6.3 Other experimental results and timings

In the table below, we list the obstructions for existence of connections in some additional cases. We used SINGULAR 3.0 on Cygwin in a Windows XP Professional (Version 2002, Service Pack 2) environment, running on a PC (AMD Athlon 64 3800+ CPU 2.41 GHz, 1 GB RAM). In one case, we were not able to calculate the obstructions due to the computational cost. The timings are the total runtimes for computations of all three obstructions.

The singularities in the table are: (1) The threefold non-Gorenstein scroll of type  $(2, 1)$ , see Yoshino (1990), proposition 16.12, (2) The threefold non-Gorenstein quotient singularity of finite CM representation type, see Yoshino (1990), proposition 16.10, (3) The cubic cone with equation  $x^3 + y^3 + z^3 = 0$ , see Laza et al. (2002), (4) The non-Gorenstein monomial curve with numerical semigroup  $(3, 4, 5)$ , and (5) The threefold  $E_6$  singularity. The modules in the table are given in the references Yoshino (1990) and Laza et al. (2002), pp. 234–235, in the first 8 cases, and as the ideals  $p = (t^3, t^4) \subseteq k[[t^3, t^4, t^5]]$  and  $m = (x, y, z, w) \subseteq k[[x, y, z, w]]/(x^3 + y^4 + z^2 + w^2)$  in the last two cases.

We remark that the module  $m$  is torsion free of depth two, while all other modules in the table are MCM, and make the following observations:

- (1) There is an isolated curve singularity with complete local ring  $A$  and an MCM  $A$ -module  $M$  such that  $M$  *does not* admit a connection.

- (2) We have not found an isolated surface singularity with complete local ring  $A$  and an MCM  $A$ -module  $M$  such that  $M$  *does not* admit a connection.
- (3) There is an isolated threefold singularity with complete local ring  $A$  and a non-free MCM  $A$ -module  $M$  such that  $M$  admits a connection.
- (4) It does not seem difficult to find an isolated singularity with complete local ring  $A$  and a (non-free) torsion free  $A$ -module  $M$  such that  $M$  admits a connection.

Singularity	Module	AClass	KSKernel	LClass	Run Time
Threefold scroll of type $(1, 2)$	$S_{-2}$	1	0	1	0.2s
	$S_{-1}$	1	0	1	0.6s
	$S_1$	1	0	1	0.6s
	$M$	1	0	1	10.6s
Threefold Q.S. of finite CM repr. type	$S_{-1}$	1	0	0	8.3s
	$M$	?	0	?	$\infty$
Cubic cone $x^3 + y^3 + z^3 = 0$	M3	1	0	0	0.9s
	M4	1	0	0	3.0s
Monomial curve	$p$	1	0	1	0.1s
Threefold $E_6$	$m$	1	0	0	0.4s

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